Children's Difficulties With Base-Ten Numbers: "Face-Value" and "Independent-Place" Constructs

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Sixteen Year 3 students, in groups of four, worked collaboratively on numeration tasks, using either conventional blocks, or computer software. Participants were interviewed individually before and after group sessions to ascertain place-value understanding. From a teaching experiment design, using a grounded theory approach, emerged previously reported categories of response, including counting approaches, grouping approaches, and "face-value" thinking. A previously unreported construct emerged, the "independent-place construct". Teachers of young children should familiarise themselves with these common approaches to place-value tasks.

The study reported on here formed the basis of the author's doctoral thesis (Price, 2001); aspects of it have previously been reported by the author (Price, 1996, 1997, 1998a, 1998b). In this paper the author introduces a previously unreported construction made by children who are not secure in their understanding of base-ten numeration, the "independent-place construct", and compares it with the "face-value construct", observed by a number of researchers.

Place value is a term used frequently in mathematical writing, referring to a central feature of the base-ten numeration system: individual digits in base-ten numbers assume a value based on their position in the entire number, relative to the rightmost place, or the place to the right of the decimal marker, where present. Unlike other numeration systems, such as the Roman and Egyptian systems, a place-value system uses a small number of symbols to represent a variety of values, depending upon the position of each symbol. This gives a place-value system a huge advantage over non-place-value systems, in being able to represent an unlimited variety of quantities. In other words, any finite amount can be represented using merely the symbols 0 to 9, with a few other symbols such as "-" and a decimal marker symbol.

Place-value understanding is arguably the most important topic in the primary mathematics curriculum. A robust understanding of place-value concepts enables a person to manage the demands of problems from a wide range of mathematical topics, especially those that involve mental computation. The importance of the development of numeracy skills in children's education is well recognised; lists of numeracy sub-skills (e.g., Queensland School Curriculum Council, 1999) typically include proficiency in use of the base-ten numeration system. However, it can seem somewhat incongruous that a system which incorporates at its heart a small number of simple rules should be so difficult for teachers to teach, and for students to learn (Jones & Thornton, 1993; Ross, 1990). Thus research into children's learning of place-value concepts has an on-going, important role in mathematics education research.

Review of the Literature

Researchers of children's mathematical understandings commonly deduce children's mental models, or "inaccessible mathematical realities" (Cobb & Steffe, 1983, p. 93) based on their performances on mathematical tasks (Davis, 1992; Resnick, 1983). The mental models of interest here are children's models for multidigit numbers; the term adopted in this study is *conceptual structures* (see also Fuson, 1990, 1992).

A number of researchers into children's development of place-value understanding have proposed schemes to describe how such understanding develops (e.g., Cobb & Wheatley, 1988; Fuson, 1990; Fuson & Briars, 1990; Fuson et al., 1997; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Resnick, 1983; Ross, 1989, 1990). Such schemes have a number of features in common. Principal among these is the idea that children pass through a number of fairly well-defined stages, as their thinking develops from early knowledge of counting number sequences to more mature understanding of multidigit numbers and how they can be used in problem solving. One such scheme, proposed by Ross (1989, 1990) is notable for the number of other authors who have either referred to it or duplicated some of Ross's methods. The author also used some of these methods, and reached a number of conclusions regarding Ross's claims, outlined in this paper.

Digit Correspondence Tasks and Face-Value Thinking.

Ross (1989) proposed a five-stage model for the development of place-value understanding, based on her study involving 60 students, 12 each from Grades 2 to 5. She assessed children's place-value understanding based on their performance on a series of tasks involving written symbols and concrete materials. Results from one task, a "digit correspondence" task, were particularly revealing: when asked to explain the referents for each digit in a two-digit number, many students responded with a "face-value" interpretation of the digits—stating that each digit in a multidigit number represented only its face value. Ross concluded from these and other results that a surprising number of students were at an early stage of development of place-value understanding, and exhibited faulty understanding of place-value rules underlying the base-ten numeration system. Digit correspondence tasks have since been used by other researchers, including Carpenter, Franke, Jacobs, Fennema, and Empson (1997); Fuson and Briars, (1990); and Miura et al., (1993), as well as in this study. Digit correspondence tasks are described in the Methods section, and results from Ross's study and this study are compared later in the paper.

Face-value thinking is possibly the most common misunderstanding of multidigit numbers held by young children, and is described in many studies reported in the literature (e.g., Bednarz & Janvier, 1982; Cobb & Wheatley, 1988; Fuson & Briars, 1990; Fuson et al., 1997; Hughes, 1995; Miura et al., 1993; Sinclair & Scheuer, 1993). This faulty thinking about multidigit numbers has been given the status of a "conceptual structure", or "construct" by some authors, who posit that it indicates a particular understanding of numbers in a place-value system held by many children.

Methods

Due to the nature of the interactions under investigation, and the necessarily imprecise view of children's thinking captured, this was an exploratory study, designed to explore children's mental models for base-ten numbers as they interacted with materials, a teacher and each other. A grounded theory approach (Strauss & Corbin, 1990) was adopted,

involving four main phases: (a) review of literature, (b) open coding of data, (c) axial coding of data, and (d) final integration of categories to form theory.

This study was conducted using a teaching experiment design, in which the researcher acted as the teacher of small groups of participants (Confrey & Lachance, 2000). A total of sixteen Year 3 students worked cooperatively in groups of four for a series of 10 sessions. Each group comprised two girls and two boys, all of whom were assessed as either of high or low mathematical achievement, based on the previous year's *Year Two Diagnostic Net* (Queensland Department of Education, 1996). The groups worked on a series of tasks designed to reinforce knowledge and understanding of two-digit numeration, and to introduce the hundreds place. Two groups used conventional place-value blocks (known to many Australian teachers as Multibase Arithmetic Blocks, or MABs) to represent numbers, as directed in various tasks. The other two groups used computer software designed by the author to represent numbers via pictures of place-value blocks which can be manipulated via various on-screen tools (see Price, 1997, for further details).

Each participant was interviewed before and after the teaching sessions, to assess any changes that occurred in participants' knowledge and understanding of place-value concepts during the study. The two interviews were parallel in form, each comprising 9 questions in five categories: number representations, skip counting, number relationships, digit correspondence, and novel tens grouping.

Digit Correspondence Tasks

As mentioned previously, digit correspondence tasks have been widely used in place-value research since they were used by Ross (1989). Digit correspondence tasks are a useful tool for investigating children's understanding of multidigit numbers, as they clearly indicate "face-value" thinking, as described later.

In a digit correspondence task, the subject is shown a collection of objects, and asked to count them. In each of the interviews used in this study, each participant was asked to count a collection of pop sticks—24 in the first interview. The participant was then asked to write the symbol for that number, and was then asked about each of the two digits "Does this part of your '24' tell you anything about how many sticks you have?" Asked this question, children with poor understanding of the base-ten numeration system will typically state that the '4' represents four sticks, and the '2' represents two sticks. Ross (1989) gave no indication that she pursued participants' understanding further in this situation; in the study described here, if a participant gave a face-value response, the interviewer asked the participant about the remaining sticks, to explain why they were not included in the sets of sticks apparently represented by the two separate digits.

In a modified digit correspondence task, also reported by other researchers, a child is provided with materials which could give misleading perceptual cues about the values represented. For example, a collection of 13 counters can be shown, and the participant asked to share them evenly among 3 cups, leaving one counter left out. In this situation, a child holding a face-value construct for two-digit numbers might be prompted to respond that the '3' in '13' stands for the three cups, and the '1' stands for the single counter left over.

During certain interview questions, including digit correspondence tasks, the researcher introduced counter-suggestions, as a means of testing the robustness of participants' beliefs. Thus, if a participant responded to a digit correspondence task to say that the tens digit represented collections of ten units, the interviewer asked if it were possible that the

digit represented only its face value. While this method does not lead to a "naturalistic" idea of what a child believes, it is felt that it can reveal a level of tentativeness in the ideas of some children.

Results

Space limitations here restrict the reporting of results from this study. In this section a focus is made on data that lead to identification of apparent face-value thinking, especially digit correspondence tasks described earlier, together with other general errors made when answering questions regarding materials, written symbols, and number names.

The Face-Value Construct

Four types of question in the interviews evoked face-value responses from participants. Briefly, these questions asked participants to: (a) compare a collection of blocks with a written symbol, such as 1 ten, 3 hundreds, and 6 ones with '136'; (b) compare written symbols with similar digits, such as '259' and '295'; (c) identify the referents for the digits in a two-digit number, after counting a collection of pop sticks; and (d) identify the referents for the digits in a two-digit number such as '13', in light of objects grouped so that there were (in this example) three collections and one unit left over.

Table 1 indicates the prevalence of responses indicating face-value thinking among the study's participants, in response to these types of interview question. Numbers in brackets indicate the number of times that participants accepted incorrect face-value suggestions from the interviewer.

Table 1

Face-Value Thinking by Participants in Response to Certain Questions

Question	Mathematical Achievement Level		Total
	High	Low	
3-digit block representations	0 (0)	5 (3)	5 (3)
Comparing pairs of numbers	0 (0)	8 (7)	8 (7)
Digit correspondence task	0 (1)	12 (0)	12 (1)
Modified digit correspondence task	3 (3)	5 (4)	8 (7)
Totals	3 (4)	30 (14)	33 (18)

Note. Values not in parentheses represent incidents of face-value interpretations initiated by participants. Values in parentheses represent face-value interpretations suggested by the researcher and accepted by participants.

It is clear from the data in Table 1 that participants of high mathematical achievement levels did not often volunteer incorrect face-value interpretations of digits, and were reluctant to accept the researcher's counter-suggestions. On the other hand, participants of low mathematical achievement levels exhibited faulty face-value thinking comparatively often. One surprising result is that the modified digit correspondence tasks, which might be expected to induce a greater number of face-value responses, did not in fact do so. Apart from responses to the interviewer's counter-suggestions, participants were more likely to

give face-value responses to the "standard" digit correspondence task, than to the one in which misleading perceptual cues were present.

Many participants gave face-value responses to digit correspondence tasks. Figure 1 illustrates a response pattern that was repeated by a number of participants when asked about the referents for the digits in the symbol '24': participants stated that the '2' represented two sticks, the '4' represented four sticks, and the remaining 18 sticks were "left-overs". It is an indication of the prevalence of face-value thinking among children of this age that, over the two interviews, 10 of the 16 participants gave responses of this type, in which they attempted to explain why there were leftover sticks when the referents for the two digits were removed from the collection. However, it is important to note that not one participant consistently used face-value explanations in answering interview questions, but participants instead exhibited a variety of ideas about numbers, with which they attempted to explain numerical situations posed to them.
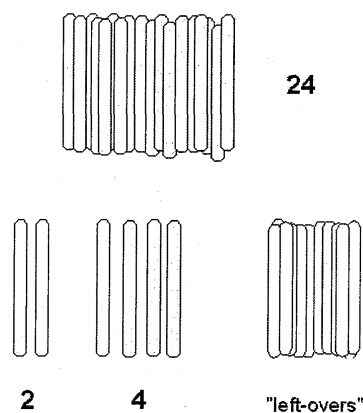

Figure 1. Typical "face-value" response to digit correspondence task.

Other Errors Made on Place-Value Tasks

The tendency of many children to explain that each digit in a multidigit number represents only its face value is just one, rather obvious, error of many made by children when dealing with multidigit numbers. In this study a large number of errors were observed. Listed below are six types of error of particular interest in light of the later discussion:

1. *Block handling* errors, such as using "incorrect" blocks to stand for each place in a multidigit number.
2. *Naming errors* of various types, such as using place names as labels only with no value attached to them.
3. *Trading errors*, such as trading a block of one size for one of another size, such as 1 ten for 1 one.
4. *Errors regarding position of blocks*, in which participants regarded the position of blocks as significant in determining the value represented by the blocks.
5. *Concatenation errors*, in which symbols were written by joining the digits for each set of blocks, such as '617' for 6 tens and 17 ones; or concatenating names, such as naming 4 tens and 12 ones as "forty-twelve".
6. *Symbol errors*, such as writing the number two hundred and four as '24'.

Discussion

The Independent-Place Construct

It would be easy to regard children's mathematical errors such those described in the previous paragraphs as signifying that their mathematical thinking is not highly developed, or to believe that their responses are unthinking, almost random responses to tasks which they cannot handle. However, a pattern emerged from this study's data that has led the author to propose the existence of another construct for multidigit numbers in the minds of many children of this age. A common element in the errors made in this study is that participants did not indicate that they perceived any link connecting "hundreds," "tens," and "ones" places, but regarded them as independent categories of quantity with separate names, separate digits, and separate block representations. The label "independent-place construct" has been coined to refer to this pattern of responses.

Evidence for the independent-place construct in this study. Each of the errors listed previously can be explained in light of an independent-place construct. For example, at various times participants were observed to choose the "wrong" block when representing a number. So, for example, Kelly was observed to use 2 ones and 8 tens to represent the number '28'. When asked if she could represent the number in a different way, she reversed the position of the same blocks, moving the 2 ones to the right and the 8 tens to the left. In another example, several participants traded a ten-block for a single one-block, indicating that they regarded the action as some sort of "block-for-block" trade, rather than a trade of equal values. Concatenation errors involved participants in joining together the names of two places represented by blocks of different sizes, as in "eighty-eleven" for 8 tens and 11 ones. When participants omitted the internal zero from a number such as '204', one interpretation is that they simply wrote one digit for each digit name encountered as they spoke the name: When asked to write '281', a child can write a digit for each number, thus: *two* [2] *hundred and eighty* [8] *-one* [1]; in the case of '204', the zero is not mentioned: *two* [2] *hundred and four* [4]. Such responses can be explained in terms of simple associations made between digits, names and blocks, with no regard for the meanings behind these representations.

The digit correspondence task responses described earlier also give support to idea that some participants were using an independent-place construct. It is possible that children giving such responses were attempting to explain the unexplainable within their knowledge of numbers: that each digit represents a finite quantity based on its face value, and that together the digits represent the entire quantity. There was evidence in the responses of some participants that they were uncomfortable with two conflicting ideas, namely that (a) each digit had a face value apparently indicating how much it represents, and (b) the sum of resulting face-value quantities does not equal the entire collection represented by the multidigit number. As two participants put it when attempting to explain the 18 sticks left out of a collection of 24 after taking out groups of 2 and 4, "Well they're nothing then if that's how that is", and "Well, they would but they're not included in ... these things". One gains the impression that the children understood more than a face-value interpretation gives them credit for, and that they were attempting to deal with an internal conflict caused by their limited knowledge of how two simple symbols represent complicated place-value ideas. Furthermore, the fact that no participant consistently used face-value responses throughout the interviews implies that the view proposed by Ross (1989) and others, that

children operate at particular levels with regards to multidigit numbers, needs to be reassessed.

Evidence for the independent-place construct in the place-value literature. There is considerable evidence in the place-value literature to support the idea of an independent-place construct. For example, Cobb and Wheatley (1988) described some of their participants as regarding tens as "abstract singletons", indivisible, separate units that could be counted apart from one units, but not as collections of 10 ones.

Several researchers (e.g., Cobb & Wheatley, 1988; Fuson & Briars, 1990; Fuson et al., 1997) have noted children adding or subtracting by considering each column separately, leading to results such as "16 + 9 = 115" and "28 + 36 = 514". Fuson et al. (1997) commented: "The physical appearance of the written multidigit marks as single digits ... may combine to seduce children to use a concatenated single-digit conceptual structure even if they have a more meaningful conception available" (p. 142). Fuson, Fraivillig, & Burghardt (1992) posed four-digit addition questions to second-graders, who proceeded to add like blocks, even though they had not previously attempted such questions. This author suggests that completing such a task is quite achievable for a student, if an independent-place construct is adopted: If each place is considered to be separate from every other, then addition of like blocks, or symbols in like columns, enables quite efficient computation of otherwise challenging questions.

Comparing face-value thinking and the independent-place construct. Though there are similarities between the face-value construct and the independent-place construct, differences between them are significant when considering how young children regard multidigit numeration tasks. It is the author's contention that the independent-place construct is more useful than the face-value construct in describing children's thinking regarding multidigit numbers, and reveals a different character to their thinking regarding base-ten numbers.

Face-value thinking, where it truly exists, is a serious obstacle to understanding the base-ten numeration system. Children who actually believe that each digit only represents its face value fail to address the core feature of a place-value numeration system, that each digit represents a certain quantity according to its position in the number. However, data in this study suggests that in at least some cases, children are instead attempting to treat each digit as separate from the others, leaving open the possibility that they have some idea that different quantities are represented, though they may be unable to manage transitions between places.

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